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$SO(4)$ symmetry of the transfer matrix for the one-dimensional Hubbard model

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Abstract. The $SO(4)$ invariance of the transfer matrix for the one-dimensional Hubbard model is clarified from the viewpoint of the quantum inverse scattering method. We demonstrate the $SO(4)$ symmetry by means of the fermionic L-operator and the fermionic R-matrix, which satisfy the graded Yang–Baxter relation. The transformation law of the fermionic L-operator under the $SO(4)$ rotation is identified with a kind of gauge transformation, which determines the corresponding transformation of the fermionic creation and annihilation operators under the $SO(4)$ rotation. The transfer matrix is confirmed to be invariant under the $SO(4)$ rotation, which ensures the $SO(4)$ invariance of the conserved currents including the Hamiltonian. Furthermore, we show that the representation of the higher conserved currents in terms of the Clifford algebra gives manifestly $SO(4)$ invariant forms.

1. Introduction

Recently, there has been much interest in the correlated electron systems. Several models are known to be exactly solvable in one dimension [1–6]. Among them the one-dimensional (1D) Hubbard model

$$\mathcal{H} = -\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c_{ms}^{\dagger}c_{m+1s} + c_{m+1s}^{\dagger}c_{ms}) + U\sum_{m=1}^{N}(n_{m\uparrow} - \tfrac{1}{2})(n_{m\downarrow} - \tfrac{1}{2}) \tag{1.1}$$

is the most important one, which describes the correlation of the electrons occupying the same site. Here $c_{ms}^{\dagger}$ and c_{ms} are the fermionic creation and annihilation operators with spin $s(=\uparrow\downarrow)$ at site $m(=1, 2, \ldots, N)$ satisfying the canonical anticommutation relations

$$\{c_{ms}^{\dagger}, c_{m's'}\} = \delta_{mm'}\delta_{ss'} \qquad \{c_{ms}^{\dagger}, c_{m's'}^{\dagger}\} = \{c_{ms}, c_{m's'}\} = 0 \tag{1.2}$$

and n_{ms} is the number density operator

$$n_{ms} = c_{ms}^{\dagger}c_{ms} \qquad (s = \uparrow\downarrow). \tag{1.3}$$

The parameter U is the coupling constant. Lieb and Wu [1] diagonalized the Hamiltonian (1.1) under the periodic boundary condition

$$c_{N+1s}^{\dagger} = c_{1s}^{\dagger} \qquad c_{N+1s} = c_{1s} \qquad (s = \uparrow\downarrow). \tag{1.4}$$

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Many physical properties have been investigated based on the associated Bethe ansatz equation (see the reprint volume [7]).

The Hamiltonian (1.1) enjoys two $su(2)$ symmetries [8–12]. One is the spin-$su(2)$ generated by

$$S^+ = \sum_{m=1}^{N} c_{m\uparrow}^\dagger c_{m\downarrow} \qquad S^- = \sum_{m=1}^{N} c_{m\downarrow}^\dagger c_{m\uparrow} \qquad S^z = \frac{1}{2}\sum_{m=1}^{N}(n_{m\uparrow} - n_{m\downarrow}) \tag{1.5}$$

and the other is charge-$su(2)$ (η-pairing $su(2)$) generated by

$$\eta^+ = \sum_{m=1}^{N}(-1)^m c_{m\uparrow}^\dagger c_{m\downarrow}^\dagger \qquad \eta^- = \sum_{m=1}^{N}(-1)^m c_{m\downarrow} c_{m\uparrow} \qquad \eta^z = \frac{1}{2}\sum_{m=1}^{N}(n_{m\uparrow} + n_{m\downarrow} - 1). \tag{1.6}$$

When we assume the periodic boundary condition (1.4), the number of sites N should be even for the consistency of the charge-$su(2)$. In this case, the spin-$su(2)$ and the charge-$su(2)$ are connected through the partial particle–hole transformation

$$c_{m\uparrow} \to c_{m\uparrow} \qquad c_{m\downarrow} \to (-1)^m c_{m\downarrow}^\dagger \qquad U \to -U. \tag{1.7}$$

As is well known, these two $su(2)$ are not independent and should be considered as elements of a bigger algebra $so(4)$ [10]

$$so(4) = su(2) \oplus su(2). \tag{1.8}$$

The $so(4)$ plays a very important role for the physical features of the 1D Hubbard model [7]. For example, it was proved by Eßler *et al* [13–15] that the Bethe ansatz states of the 1D Hubbard model are incomplete and have to be complemented by the $so(4)$ symmetry. Eßler and Korepin [16, 17] showed that the elementary excitations of the half-filled band constitute the multiplets of $so(4)$.

Several authors have discussed the generalization of the Lie algebra symmetry $so(4)$ to the group symmetry $SO(4)$. Following Affleck *et al* [18], we introduce the 2×2 matrices

$$\Psi_{2n-1} = \begin{pmatrix} c_{2n-1\downarrow}^\dagger & \mathrm{i}c_{2n-1\uparrow} \\ \mathrm{i}c_{2n-1\uparrow}^\dagger & c_{2n-1\downarrow} \end{pmatrix} \qquad \Psi_{2n} = \begin{pmatrix} c_{2n\downarrow}^\dagger & -\mathrm{i}c_{2n\uparrow} \\ \mathrm{i}c_{2n\uparrow}^\dagger & -c_{2n\downarrow} \end{pmatrix} \qquad n = 1, \ldots, \frac{N}{2}. \tag{1.9}$$

For convenience, the definition of Ψ_m in this paper is chosen to be different from the usual one [12, 18]. However, they are essentially equivalent.

The spin-$SU(2)$ transformation can be realized by the left multiplication of an $SU(2)$ matrix

$$\Psi_m \longrightarrow \mathcal{O}_{\mathrm{spin}}\Psi_m \qquad \mathcal{O}_{\mathrm{spin}} \in SU(2)$$

while the charge-$SU(2)$ transformation corresponds to the right multiplication of another $SU(2)$ matrix,

$$\Psi_m \longrightarrow \Psi_m\mathcal{O}_{\mathrm{charge}} \qquad \mathcal{O}_{\mathrm{charge}} \in SU(2).$$

Since the left and the right matrix multiplications are commutative, the transformation

$$\tilde{\Psi}_m = \mathcal{O}_{\mathrm{spin}}\Psi_m\mathcal{O}_{\mathrm{charge}} \tag{1.10}$$

gives the $SU(2) \times SU(2)$ transformation among the fermion operators. More precisely, the exact group symmetry is

$$SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2$$

because the choices $\mathcal{O}_{\text{spin}} = -\mathbf{1}$, $\mathcal{O}_{\text{charge}} = \mathbf{1}$ and $\mathcal{O}_{\text{spin}} = \mathbf{1}$, $\mathcal{O}_{\text{charge}} = -\mathbf{1}$ induce the same transformation. The infinitesimal transformation of (1.10) gives the Lie algebra symmetry (1.8).

The integrability of the 1D Hubbard model with the periodic boundary condition was established by Shastry [19–21] and Olmedilla and co-workers [22, 23]. Shastry introduced a Jordan–Wigner transformation, which changes the fermionic Hamiltonian (1.1) into an equivalent coupled spin model

$$H = \sum_{m=1}^{N}(\sigma_{m+1}^{+}\sigma_{m}^{-} + \sigma_{m}^{+}\sigma_{m+1}^{-}) + \sum_{m=1}^{N}(\tau_{m+1}^{+}\tau_{m}^{-} + \tau_{m}^{+}\tau_{m+1}^{-}) + \frac{U}{4}\sum_{m=1}^{N}\sigma_{m}^{z}\tau_{m}^{z}. \tag{1.11}$$

Here σ and τ are two species of the Pauli matrices commuting each other. For this equivalent coupled spin model, Shastry constructed the L-operator and the R-matrix (see the appendix), which satisfy the Yang–Baxter relation

$$\check{R}_{12}(\theta_1, \theta_2)[L_m(\theta_1) \otimes L_m(\theta_2)] = [L_m(\theta_2) \otimes L_m(\theta_1)]\check{R}_{12}(\theta_1, \theta_2). \tag{1.12}$$

The Yang–Baxter equation for Shastry's R-matrix was recently proved in [24] (see also [25, 26]).

The coupled spin model (1.11) is also referred to as the 1D Hubbard model, since they are related through the Jordan–Wigner transformation. However, there are differences between the coupled spin model (1.11) and the fermionic Hamiltonian (1.1). It is well known that the periodic boundary condition for the fermion model does not correspond to the periodic boundary condition for the equivalent spin model. Moreover, due to the non-locality of the Jordan–Wigner transformation, the generators of the $so(4)$ symmetry, (1.5) and (1.6), become non-local in terms of the spin operators σ and τ. Thus, it is more transparent to employ the fermionic formulation of the Yang–Baxter relation developed by Olmedilla *et al* [22], when we investigate the $SO(4)$ or other symmetries of the 1D Hubbard model from the viewpoint of the quantum inverse scattering method (QISM).

Recently, Göhmann and Murakami [27] demonstrated that the transfer matrix constructed from the fermionic L-operators has the $su(2) \oplus su(2)$ symmetry. The main purpose of this paper is to generalize their result to the finite symmetry, namely the $SO(4)$ symmetry corresponding to (1.10).

Compared with other models (see [29–31]), the algebraic structure of the 1D Hubbard model is not yet fully clarified. Our result will provide a further step to the complete understanding of the mathematical structure of the model.

This paper is organized as follows. In section 2, we give a brief summary of the fermionic formulation of the QISM for the 1D Hubbard model. Some important properties of the fermionic R-matrix are explained. In section 3, we prove the $SO(4)$ invariance of the fermionic transfer matrix. It is shown that the $SO(4)$ rotation for the fermion operators is related to a kind of gauge transformation of the fermionic L-operator. When the number of sites is even, we can establish the $SO(4)$ symmetry of the transfer matrix under the periodic boundary condition. When the number of sites is odd, we have to impose a twisted periodic boundary condition to establish the $SO(4)$ symmetry of the transfer matrix. In section 4, we discuss the invariance of the transfer matrix under the partial particle–hole transformation. In section 5, we give a new representation of some higher conserved currents using the Clifford algebra. The $SO(4)$ invariance of the conserved currents becomes obvious in this representation. The final section is devoted to discussions.

2. Graded Yang–Baxter relation for the 1D Hubbard model

As a preparation for later sections, we shall summarize the fermionic formulation of the 1D Hubbard model [22, 23, 28]. The fermionic L-operator is

$$
\mathcal{L}_m(\theta) = \begin{pmatrix} -\mathrm{e}^{h} f_{m\uparrow}(\theta) f_{m\downarrow}(\theta) & -f_{m\uparrow}(\theta) c_{m\downarrow} & \mathrm{i} c_{m\uparrow} f_{m\downarrow}(\theta) & \mathrm{i}\mathrm{e}^{h} c_{m\uparrow} c_{m\downarrow} \\ -\mathrm{i} f_{m\uparrow}(\theta) c^{\dagger}_{m\downarrow} & \mathrm{e}^{-h} f_{m\uparrow}(\theta) g_{m\downarrow}(\theta) & \mathrm{e}^{-h} c_{m\uparrow} c^{\dagger}_{m\downarrow} & \mathrm{i} c_{m\uparrow} g_{m\downarrow}(\theta) \\ c^{\dagger}_{m\uparrow} f_{m\downarrow}(\theta) & \mathrm{e}^{-h} c^{\dagger}_{m\uparrow} c_{m\downarrow} & \mathrm{e}^{-h} g_{m\uparrow}(\theta) f_{m\downarrow}(\theta) & g_{m\uparrow}(\theta) c_{m\downarrow} \\ -\mathrm{i}\mathrm{e}^{h} c^{\dagger}_{m\uparrow} c^{\dagger}_{m\downarrow} & c^{\dagger}_{m\uparrow} g_{m\downarrow}(\theta) & \mathrm{i} g_{m\uparrow}(\theta) c^{\dagger}_{m\downarrow} & -\mathrm{e}^{h} g_{m\uparrow}(\theta) g_{m\downarrow}(\theta) \end{pmatrix} \tag{2.1}
$$

where

$$
\begin{aligned} f_{ms}(\theta) &= \sin\theta - \{\sin\theta - \mathrm{i}\cos\theta\} n_{ms} \\ g_{ms}(\theta) &= \cos\theta - \{\cos\theta + \mathrm{i}\sin\theta\} n_{ms}. \end{aligned}
$$

The parameter h is related to the spectral parameter θ and the Coulomb coupling constant U through the relation

$$
\frac{\sinh 2h}{\sin 2\theta} = \frac{U}{4}. \tag{2.2}
$$

We express by $\underset{s}{\otimes}$ the Grassmann (graded) direct product

$$
\begin{aligned} &[A \underset{s}{\otimes} B]_{\alpha\gamma,\beta\delta} = (-1)^{[P(\alpha)+P(\beta)]P(\gamma)} A_{\alpha\beta} B_{\gamma\delta} \\ &P(1) = P(4) = 0 \qquad P(2) = P(3) = 1. \end{aligned} \tag{2.3}
$$

The fermionic L-operator satisfies the graded Yang–Baxter relation [22]

$$
\check{\mathcal{R}}_{12}(\theta_1,\theta_2)[\mathcal{L}_m(\theta_1) \underset{s}{\otimes} \mathcal{L}_m(\theta_2)] = [\mathcal{L}_m(\theta_2) \underset{s}{\otimes} \mathcal{L}_m(\theta_1)]\check{\mathcal{R}}_{12}(\theta_1,\theta_2) \tag{2.4}
$$

under the constraint of the spectral parameter (2.2).

The explicit form of the fermionic R-matrix is [22]

$$
\check{\mathcal{R}}_{12}(\theta_1,\theta_2) = \begin{pmatrix}
a^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e & 0 & 0 & \mathrm{i}b^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e & 0 & 0 & 0 & 0 & 0 & \mathrm{i}b^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d^+ & 0 & 0 & -\mathrm{i}f & 0 & 0 & \mathrm{i}f & 0 & 0 & -c^+ & 0 & 0 & 0 \\
0 & -\mathrm{i}b^+ & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a^- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{i}f & 0 & 0 & d^- & 0 & 0 & -c^- & 0 & 0 & -\mathrm{i}f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}b^+ & 0 & 0 \\
0 & 0 & -\mathrm{i}b^+ & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{i}f & 0 & 0 & -c^- & 0 & 0 & d^- & 0 & 0 & \mathrm{i}f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & -\mathrm{i}b^+ & 0 \\
0 & 0 & 0 & -c^+ & 0 & 0 & \mathrm{i}f & 0 & 0 & -\mathrm{i}f & 0 & 0 & d^+ & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}b^- & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}b^- & 0 & 0 & e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^+
\end{pmatrix} \tag{2.5}
$$

where

$$
\begin{aligned}
a^{\pm} &= \cos^2(\theta_1-\theta_2)\left\{1 \pm \tanh(h_1-h_2)\frac{\cos(\theta_1+\theta_2)}{\cos(\theta_1-\theta_2)}\right\}\\
b^{\pm} &= \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)\left\{1 \pm \tanh(h_1-h_2)\frac{\sin(\theta_1+\theta_2)}{\sin(\theta_1-\theta_2)}\right\}\\
&= \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)\left\{1 \pm \tanh(h_1+h_2)\frac{\cos(\theta_1+\theta_2)}{\cos(\theta_1-\theta_2)}\right\}\\
c^{\pm} &= \sin^2(\theta_1-\theta_2)\left\{1 \pm \tanh(h_1+h_2)\frac{\sin(\theta_1+\theta_2)}{\sin(\theta_1-\theta_2)}\right\}\\
d^{\pm} &= 1 \pm \tanh(h_1-h_2)\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)} = 1 \pm \tanh(h_1+h_2)\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}\\
e &= \frac{\cos(\theta_1-\theta_2)}{\cosh(h_1-h_2)} \qquad f = \frac{\sin(\theta_1-\theta_2)}{\cosh(h_1+h_2)}.
\end{aligned}
\tag{2.6}
$$

The second equalities for the Boltzmann weights $b^{\pm}$ and $d^{\pm}$ are valid due to constraint (2.2).

For convenience, we introduce an equivalent fermionic R-matrix

$$\mathcal{R}_{12}(\theta_1,\theta_2) \equiv \mathcal{P}_{12}\check{\mathcal{R}}_{12}(\theta_1,\theta_2) \tag{2.7}$$

where $\mathcal{P}_{12}$ is the graded permutation

$$\mathcal{P}_{\alpha\gamma,\beta\delta} = (-1)^{P(\alpha)P(\gamma)}\delta_{\alpha\delta}\delta_{\gamma\beta}. \tag{2.8}$$

The fundamental properties of the fermionic R-matrix $\mathcal{R}_{12}(\theta_1,\theta_2)$ are summarized as follows [32].

(1) Regularity (initial condition):

$$\mathcal{R}_{12}(\theta_0,\theta_0) = \mathcal{P}_{12}. \tag{2.9}$$

(2) Graded Yang–Baxter equation:

$$\mathcal{R}_{12}(\theta_1,\theta_2)\mathcal{R}_{13}(\theta_1,\theta_3)\mathcal{R}_{23}(\theta_2,\theta_3) = \mathcal{R}_{23}(\theta_2,\theta_3)\mathcal{R}_{13}(\theta_1,\theta_3)\mathcal{R}_{12}(\theta_1,\theta_2) \tag{2.10}$$

under the constraints

$$\frac{\sinh 2h_j}{\sin 2\theta_j} = \frac{U}{4} \qquad j = 1, 2, 3.$$

(3) Unitarity:

$$\mathcal{R}_{12}(\theta_1,\theta_2)\mathcal{R}_{21}(\theta_2,\theta_1) = \rho(\theta_1,\theta_2)I \tag{2.11}$$

where

$$\mathcal{R}_{21}(\theta_2,\theta_1) \equiv \mathcal{P}_{12}\mathcal{R}_{12}(\theta_2,\theta_1)\mathcal{P}_{12}$$

and

$$\rho(\theta_1,\theta_2) = \cos^2(\theta_1-\theta_2)[\cos^2(\theta_1-\theta_2) - \tanh^2(h_1-h_2)\cos^2(\theta_1+\theta_2)].$$

Since the non-zero elements of the R-matrix (2.7) are even with respect to the parity $P(\alpha)$ (2.3), i.e.

$$P(\alpha)+P(\beta)+P(\alpha')+P(\beta') = 0 \pmod 2 \qquad \text{for } \mathcal{R}_{\alpha\beta;\alpha'\beta'}(\theta_1,\theta_2) \neq 0$$

the graded Yang–Baxter equation (2.10) can be expressed in terms of the matrix elements as [33]

$$\mathcal{R}_{\alpha\beta;\alpha''\beta''}(\theta_1,\theta_2)\mathcal{R}_{\alpha''\gamma;\alpha'\gamma''}(\theta_1,\theta_3)\mathcal{R}_{\beta''\gamma'';\beta'\gamma'}(\theta_2,\theta_3)(-)^{P(\beta'')[P(\alpha')+P(\alpha'')]}$$
$$= \mathcal{R}_{\beta\gamma;\beta''\gamma''}(\theta_2,\theta_3)\mathcal{R}_{\alpha\gamma'';\alpha''\gamma'}(\theta_1,\theta_3)\mathcal{R}_{\alpha''\beta'';\alpha'\beta'}(\theta_1,\theta_2)(-)^{P(\beta'')[P(\alpha)+P(\alpha'')]}. \quad (2.12)$$

Here the summations are taken over the repeated indices.

In our previous work [32], we found two important relations of the fermionic R-matrix with constant matrices M and V. The first relation is the symmetry of the fermionic R-matrix

$$\left[\check{\mathcal{R}}_{12}(\theta_1,\theta_2), \mathsf{M} \underset{s}{\otimes} \mathsf{M}\right] = 0 \quad (2.13)$$

where the general form of the matrix M is given by

$$\mathsf{M} = \begin{pmatrix} M_{11} & 0 & 0 & M_{14} \\ 0 & M_{22} & M_{23} & 0 \\ 0 & M_{32} & M_{33} & 0 \\ M_{41} & 0 & 0 & M_{44} \end{pmatrix} \quad (2.14)$$

with the condition

$$\Delta\mathsf{M} \equiv M_{11}M_{44} - M_{41}M_{14} = M_{22}M_{33} - M_{23}M_{32}. \quad (2.15)$$

We call the matrix M symmetry matrix. For simplicity, we assume $\Delta\mathsf{M} = 1$ throughout the paper.

The second relation is

$$\check{\mathcal{R}}_{12}(\theta_1,\theta_2;U)\left[\mathsf{V} \underset{s}{\otimes} \mathsf{V}\right] = \left[\mathsf{V} \underset{s}{\otimes} \mathsf{V}\right]\check{\mathcal{R}}_{12}(\theta_1,\theta_2;-U) \quad (2.16)$$

where the general form of the matrix V is given by

$$\mathsf{V} = \begin{pmatrix} 0 & V_{12} & V_{13} & 0 \\ V_{21} & 0 & 0 & V_{24} \\ V_{31} & 0 & 0 & V_{34} \\ 0 & V_{42} & V_{43} & 0 \end{pmatrix}$$
$$V_{12}V_{43} - V_{13}V_{42} = V_{21}V_{34} - V_{31}V_{24}. \quad (2.17)$$

In relation (2.16), the U-dependence of the fermionic R-matrix is explicitly written. The coupling constant of the fermionic R-matrix on the RHS is $-U$, or equivalently, $h_1 \to -h_1$ and $h_2 \to -h_2$. The matrix V is related to the partial particle–hole transformation (1.7). The constant matrices M and V play an important role in the consideration of the symmetry of the transfer matrix for the 1D Hubbard model (see section 3 and section 4). We remark that the symmetry matrix of Shastry's R-matrix is not of the form (2.14) (see the appendix). This gives one of the reasons why the fermionic formulation employed in this paper is more appropriate for the investigation of the 1D Hubbard model (1.1).

The monodromy matrix is defined as the ordered product of the fermionic L-operators

$$T(\theta) = \prod_{m=1}^{\overleftarrow{N}} \mathcal{L}_m(\theta) = \mathcal{L}_N(\theta)\cdots\mathcal{L}_1(\theta). \quad (2.18)$$

From the (local) graded Yang–Baxter relation (2.4), we have the global relation for the monodromy matrix

$$\check{\mathcal{R}}_{12}(\theta_1,\theta_2)[T(\theta_1) \underset{s}{\otimes} T(\theta_2)] = [T(\theta_2) \underset{s}{\otimes} T(\theta_1)]\check{\mathcal{R}}_{12}(\theta_1,\theta_2). \quad (2.19)$$

Define the (fermionic) transfer matrix by

$$\mathrm{str}\mathbf{K}T(\theta) \equiv \mathrm{tr}[(\sigma^z \otimes \sigma^z)\mathbf{K}T(\theta)] \tag{2.20}$$

where the constant matrix $\mathbf{K}$ assumes the form (2.14) and determines the boundary condition [32]. In particular, $\mathbf{K} = I$ corresponds to the periodic boundary condition (1.4). Then from the global graded Yang–Baxter relation (2.19), we can prove that the transfer matrix (2.20) constitutes a commuting family

$$[\mathrm{str}\mathbf{K}T(\theta_1), \mathrm{str}\mathbf{K}T(\theta_2)] = 0 \tag{2.21}$$

which proves the integrability of the 1D Hubbard model with the (twisted) periodic boundary condition.

3. $SO(4)$ symmetry of the fermionic transfer matrix

We shall discuss the $SO(4)$ symmetry of the fermionic transfer matrix (2.20). Let us consider the following transformation of the fermionic L-operator

$$\tilde{\mathcal{L}}_m(\theta) = \bar{\mathbf{M}}^{-1}\mathcal{L}_m(\theta)\mathbf{M} \tag{3.1}$$

where the constant matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ have the form of the symmetry matrix (2.14).

Since the matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ are the symmetry matrices, the transformed L-operator $\tilde{\mathcal{L}}_m(\theta)$ (3.1) also satisfies the graded Yang–Baxter relation with the *same* fermionic R-matrix

$$\check{\mathcal{R}}_{12}(\theta_1,\theta_2)[\tilde{\mathcal{L}}_m(\theta_1) \underset{s}{\otimes} \tilde{\mathcal{L}}_m(\theta_2)] = [\tilde{\mathcal{L}}_m(\theta_2) \underset{s}{\otimes} \tilde{\mathcal{L}}_m(\theta_1)]\check{\mathcal{R}}_{12}(\theta_1,\theta_2). \tag{3.2}$$

We now look for a special transformation of (3.1), which satisfies

$$\tilde{\mathcal{L}}_m(\theta; c_{ms}) = \mathcal{L}_m(\theta; \tilde{c}_{ms}). \tag{3.3}$$

Here we explicitly write the dependence of the fermionic L-operator on the fermion operators. The fermion operators c_{ms} and $\tilde{c}_{ms}$ are assumed to be connected through the transformation law (1.10). We discovered that relation (3.3) is satisfied when the matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ meet the following conditions

$$\begin{array}{llll} M_{44} = M_{11}^* & M_{41} = -M_{14}^* & M_{33} = M_{22}^* & M_{32} = -M_{23}^* \\ \bar{M}_{11} = M_{11} & \bar{M}_{44} = M_{44} & \bar{M}_{14} = -M_{14} & \bar{M}_{41} = -M_{41} \\ \bar{M}_{22} = M_{22} & \bar{M}_{33} = M_{33} & \bar{M}_{23} = M_{23} & \bar{M}_{32} = M_{32}. \end{array} \tag{3.4}$$

Condition (2.15) now becomes

$$|M_{11}|^2 + |M_{14}|^2 = |M_{22}|^2 + |M_{23}|^2 = 1. \tag{3.5}$$

It is useful to introduce the submatrices of the matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ as

$$\begin{array}{ll} \mathbf{M}_{\mathrm{charge}} = \begin{pmatrix} M_{11} & M_{14} \\ M_{41} & M_{44} \end{pmatrix} & \mathbf{M}_{\mathrm{spin}} = \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix} \\ \bar{\mathbf{M}}_{\mathrm{charge}} = \begin{pmatrix} \bar{M}_{11} & \bar{M}_{14} \\ \bar{M}_{41} & \bar{M}_{44} \end{pmatrix} & \bar{\mathbf{M}}_{\mathrm{spin}} = \begin{pmatrix} \bar{M}_{22} & \bar{M}_{23} \\ \bar{M}_{32} & \bar{M}_{33} \end{pmatrix}. \end{array} \tag{3.6}$$

Then conditions (3.4) and (3.5) are equivalent to the relations

$$\bar{\mathbf{M}}_{\mathrm{charge}} = \sigma^z\mathbf{M}_{\mathrm{charge}}\sigma^z \qquad \bar{\mathbf{M}}_{\mathrm{spin}} = \mathbf{M}_{\mathrm{spin}} \qquad \mathbf{M}_{\mathrm{charge}}, \mathbf{M}_{\mathrm{spin}} \in SU(2). \tag{3.7}$$

The corresponding transformation law of the fermion operators is

$$\begin{pmatrix} \tilde{c}_{m\downarrow}^\dagger & \mathrm{i}\tilde{c}_{m\uparrow} \\ \mathrm{i}\tilde{c}_{m\uparrow}^\dagger & \tilde{c}_{m\downarrow} \end{pmatrix} = \begin{pmatrix} M_{22}^* & -M_{23} \\ M_{23}^* & M_{22} \end{pmatrix} \begin{pmatrix} c_{m\downarrow}^\dagger & \mathrm{i}c_{m\uparrow} \\ \mathrm{i}c_{m\uparrow}^\dagger & c_{m\downarrow} \end{pmatrix} \begin{pmatrix} M_{11} & M_{14} \\ -M_{14}^* & M_{11}^* \end{pmatrix}. \tag{3.8}$$

Hereafter, we implicitly assume conditions (3.7) for matrices M and $\bar{\mathsf{M}}$. Then transformation (3.1) is not a gauge transformation in a strict sense, because $\bar{\mathsf{M}} \neq \mathsf{M}$ (particularly $\bar{\mathsf{M}}_{\text{charge}} \neq \mathsf{M}_{\text{charge}}$). So we try assigning the different transformation laws to the L-operators for odd and even sites as

$$\tilde{\mathcal{L}}_{2n-1}(\theta) = \bar{\mathsf{M}}^{-1}\mathcal{L}_{2n-1}(\theta)\mathsf{M} \qquad \tilde{\mathcal{L}}_{2n}(\theta) = \mathsf{M}^{-1}\mathcal{L}_{2n}(\theta)\bar{\mathsf{M}}. \tag{3.9}$$

The corresponding transformation law of the fermion operators on odd sites is, of course, given by formula (3.8)

$$\begin{pmatrix} \tilde{c}^{\dagger}_{2n-1\downarrow} & \mathrm{i}\tilde{c}_{2n-1\uparrow} \\ \mathrm{i}\tilde{c}^{\dagger}_{2n-1\uparrow} & \tilde{c}_{2n-1\downarrow} \end{pmatrix} = \begin{pmatrix} M^{*}_{22} & -M_{23} \\ M^{*}_{23} & M_{22} \end{pmatrix} \begin{pmatrix} c^{\dagger}_{2n-1\downarrow} & \mathrm{i}c_{2n-1\uparrow} \\ \mathrm{i}c^{\dagger}_{2n-1\uparrow} & c_{2n-1\downarrow} \end{pmatrix} \begin{pmatrix} M_{11} & M_{14} \\ -M^{*}_{14} & M^{*}_{11} \end{pmatrix}. \tag{3.10}$$

Using the fact that the matrices M and $\bar{\mathsf{M}}$ are related by the exchange $M_{14} \leftrightarrow -M_{14}$, the transformation law for even sites can readily obtained as

$$\begin{pmatrix} \tilde{c}^{\dagger}_{2n\downarrow} & -\mathrm{i}\tilde{c}_{2n\uparrow} \\ \mathrm{i}\tilde{c}^{\dagger}_{2n\uparrow} & -\tilde{c}_{2n\downarrow} \end{pmatrix} = \begin{pmatrix} M^{*}_{22} & -M_{23} \\ M^{*}_{23} & M_{22} \end{pmatrix} \begin{pmatrix} c^{\dagger}_{2n\downarrow} & -\mathrm{i}c_{2n\uparrow} \\ \mathrm{i}c^{\dagger}_{2n\uparrow} & -c_{2n\downarrow} \end{pmatrix} \begin{pmatrix} M_{11} & M_{14} \\ -M^{*}_{14} & M^{*}_{11} \end{pmatrix}. \tag{3.11}$$

Recalling the definition of the 2×2 matrices Ψ_m (1.9), we can summarize the transformation laws (3.10) and (3.11) as

$$\tilde{\Psi}_m = \mathsf{M}^{-1}_{\text{spin}}\Psi_m\mathsf{M}_{\text{charge}} \tag{3.12}$$

which exactly coincides (1.10) with the correspondences $\mathcal{O}_{\text{spin}} = \mathsf{M}^{-1}_{\text{spin}}$ and $\mathcal{O}_{\text{charge}} = \mathsf{M}_{\text{charge}}$. As we explained in section 1, transformation (3.12) is the $SO(4)$ rotation in the space of the fermion operators. Therefore we can conclude that a kind of gauge transformation (3.9) induces the $SO(4)$ rotations for the fermion operators. We call transformation (3.9) the $SO(4)$ rotation for the fermionic L-operator (2.1). Note that the canonical anticommutation relation (1.2) is preserved under transformation (3.12)

$$\{\tilde{c}^{\dagger}_{ms}, \tilde{c}_{m's'}\} = \delta_{mm'}\delta_{ss'} \qquad \{\tilde{c}^{\dagger}_{ms}, \tilde{c}^{\dagger}_{m's'}\} = \{\tilde{c}_{ms}, \tilde{c}_{m's'}\} = 0.$$

Let us consider the $SO(4)$ invariance of the fermionic transfer matrix (2.20). First we assume that N is even and impose the periodic boundary condition. The local $SO(4)$ rotation for the fermionic L-operators (3.9) induces the $SO(4)$ rotation for the monodromy matrix (2.18)

$$\tilde{T}(\theta) \equiv \overset{\longleftarrow}{\prod_{m=1}^{N}} \tilde{\mathcal{L}}_m(\theta) = \mathsf{M}^{-1}T(\theta)\mathsf{M}. \tag{3.13}$$

Since the relation

$$\operatorname{str}\{X(\theta)\mathsf{M}\} = \operatorname{str}\{\mathsf{M}X(\theta)\} \tag{3.14}$$

holds, the transfer matrix (2.20) is invariant under the periodic boundary condition ($\mathsf{K} = I$)

$$\operatorname{str}\tilde{T}(\theta; c_{ms}) = \operatorname{str}T(\theta; c_{ms}) \tag{3.15}$$

where we write the fermion operators explicitly. In relation (3.14), $X(\theta)$ is any 4×4 matrix, which may depend on the fermion operators. On the other hand, the transfer matrix $\operatorname{str}\tilde{T}(\theta; c_{ms})$ can be expressed as

$$\operatorname{str}\tilde{T}(\theta; c_{ms}) = \operatorname{str}T(\theta; \tilde{c}_{ms}) \tag{3.16}$$

due to property (3.3). Combining (3.15) and (3.16), we establish

$$\operatorname{str}T(\theta; c_{ms}) = \operatorname{str}T(\theta; \tilde{c}_{ms}). \tag{3.17}$$

Relation (3.17) shows that the fermionic transfer matrix is invariant under the $SO(4)$ rotation for the fermion operators (3.12). It indicates that all the higher conserved currents, which are embedded in the transfer matrix, also have the $SO(4)$ symmetry (see section 5).

We shall now write transformation (3.13) in terms of the submatrices $\mathsf{M}_{\text{charge}}$ and M_{spin} (3.6). We introduce the following convenient notation for the monodromy matrix [27]

$$T(\theta)=\begin{pmatrix} D_{11}(\theta) & C_{11}(\theta) & C_{12}(\theta) & D_{12}(\theta) \\ B_{11}(\theta) & A_{11}(\theta) & A_{12}(\theta) & B_{12}(\theta) \\ B_{21}(\theta) & A_{21}(\theta) & A_{22}(\theta) & B_{22}(\theta) \\ D_{21}(\theta) & C_{21}(\theta) & C_{22}(\theta) & D_{22}(\theta) \end{pmatrix}$$

where we regard $A(\theta)=(A_{ij}(\theta))$, $B(\theta)=(B_{ij}(\theta))$, $C(\theta)=(C_{ij}(\theta))$ and $D(\theta)=(D_{ij}(\theta))$ as 2×2 matrices. Then transformation (3.13) can be expressed in terms of 2×2 matrices $A(\theta),\ldots,D(\theta)$ as

$$\begin{aligned} \tilde{A}(\theta) &= \mathsf{M}_{\text{spin}}^{-1}A(\theta)\mathsf{M}_{\text{spin}} \\ \tilde{B}(\theta) &= \mathsf{M}_{\text{spin}}^{-1}B(\theta)\mathsf{M}_{\text{charge}} \\ \tilde{C}(\theta) &= \mathsf{M}_{\text{charge}}^{-1}C(\theta)\mathsf{M}_{\text{spin}} \\ \tilde{D}(\theta) &= \mathsf{M}_{\text{charge}}^{-1}D(\theta)\mathsf{M}_{\text{charge}}. \end{aligned} \tag{3.18}$$

Because the transformed monodromy matrix also satisfies the graded Yang–Baxter relation with the fermionic R-matrix

$$\check{R}_{12}(\theta_1,\theta_2)\Big[\tilde{T}(\theta_1)\underset{s}{\otimes}\tilde{T}(\theta_2)\Big]=\Big[\tilde{T}(\theta_2)\underset{s}{\otimes}\tilde{T}(\theta_1)\Big]\check{R}_{12}(\theta_1,\theta_2)$$

the associative algebra defined by the graded Yang–Baxter relation should be invariant under transformation (3.18). From (3.18), we notice an interesting fact that the submatrix $A(\theta)$ is transformed by the spin-$SU(2)$ rotation and $D(\theta)$ is transformed by the charge-$SU(2)$ rotation. We believe that this property plays a significant role in the application of the algebraic Bethe ansatz for the 1D Hubbard model [34].

Next we consider the case of N odd. The monodromy matrix transforms as

$$\tilde{T}(\theta)=\bar{\mathsf{M}}^{-1}T(\theta)\mathsf{M}. \tag{3.19}$$

In this case, we have to twist the periodic boundary condition to make the transfer matrix $SO(4)$ invariant. The condition for the matrix K in the transfer matrix is

$$\mathsf{K}\bar{\mathsf{M}}=\mathsf{M}\mathsf{K}. \tag{3.20}$$

For example

$$\mathsf{K}=\begin{pmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \tag{3.21}$$

solves the condition (3.20). From the formula in [32] (equation (3.17)), we can see that the choice (3.21) corresponds to the twisted boundary condition

$$\begin{aligned} c_{N+1\uparrow}^{\dagger} &= \mathrm{i}c_{1\uparrow}^{\dagger} \qquad & c_{N+1\uparrow} &= -\mathrm{i}c_{1\uparrow} \\ c_{N+1\downarrow}^{\dagger} &= \mathrm{i}c_{1\downarrow}^{\dagger} \qquad & c_{N+1\downarrow} &= -\mathrm{i}c_{1\downarrow}. \end{aligned} \tag{3.22}$$

Assuming (3.21) and (3.22), we can prove the $SO(4)$ invariance of the transfer matrix for N odd as

$$\mathrm{str}\mathsf{K}T(\theta;c_{ms})=\mathrm{str}\mathsf{K}T(\theta;\tilde{c}_{ms}) \tag{3.23}$$

in a similar way to the even case.

4. Partial particle–hole transformation of the fermionic transfer matrix

The transformation law of the fermionic L-operator corresponding to the partial particle–hole transformation (1.7) was found in [27]. We shall discuss the transformation law in connection with the relation (2.16). Consider the following transformations of the fermionic L-operators

$$\hat{\mathcal{L}}_{2n-1}(\theta) = \bar{\mathbf{V}}^{-1}\mathcal{L}_{2n-1}(\theta)\mathbf{V} \qquad \hat{\mathcal{L}}_{2n}(\theta) = \mathbf{V}^{-1}\mathcal{L}_{2n}(\theta)\bar{\mathbf{V}} \tag{4.1}$$

where

$$\mathbf{V} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \bar{\mathbf{V}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.2}$$

Since the constant matrices $\mathbf{V}$ and $\bar{\mathbf{V}}$ are of the form (2.17), we have the following graded Yang–Baxter relation with the transformed L-operator (4.1)

$$\check{\mathcal{R}}_{12}(\theta_1,\theta_2;-U)[\hat{\mathcal{L}}_m(\theta_1;U)\underset{s}{\otimes}\hat{\mathcal{L}}_m(\theta_2;U)] = [\hat{\mathcal{L}}_m(\theta_2;U)\underset{s}{\otimes}\hat{\mathcal{L}}_m(\theta_1;U)]\check{\mathcal{R}}_{12}(\theta_1,\theta_2;-U) \tag{4.3}$$

where we write the U-dependence explicitly. The graded Yang–Baxter relation (4.3) implies that the transformed L-operators $\hat{\mathcal{L}}_m(\theta;U)$ are related to the L-operators with the coupling constant $-U$. In fact the following relations hold

$$\begin{aligned} &\hat{\mathcal{L}}_{2n-1}(\theta; c_{2n-1s}, U) = \mathrm{i}\mathcal{L}_{2n-1}(\theta; \hat{c}_{2n-1s}, -U) \\ &\hat{\mathcal{L}}_{2n}(\theta; c_{2ns}, U) = \mathrm{i}\mathcal{L}_{2n}(\theta; \hat{c}_{2ns}, -U) \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &\hat{c}_{2n-1\uparrow} = c_{2n-1\uparrow} \qquad \hat{c}_{2n-1\downarrow} = -c^{\dagger}_{2n-1\downarrow} \\ &\hat{c}_{2n\uparrow} = c_{2n\uparrow} \qquad \hat{c}_{2n\downarrow} = c^{\dagger}_{2n\downarrow}. \end{aligned} \tag{4.5}$$

Transformation (4.5) is nothing but the partial particle–hole transformation (1.7). Therefore we call (4.1) the partial particle–hole transformation of the fermionic L-operator (2.1).

It is quite interesting to note that transformation (4.5) can be written in terms of the 2×2 matrix Ψ_m (1.9) as

$$\hat{\Psi}_m = (-1)^m \Psi_m^{\dagger} \tag{4.6}$$

where † denotes the Hermitian conjugation. Moreover, taking the Hermitian conjugation of (3.12), we find

$$\tilde{\hat{\Psi}}_m = \mathbf{M}_{\text{charge}}^{-1}\hat{\Psi}_m\mathbf{M}_{\text{spin}} \tag{4.7}$$

which shows that the spin-$SU(2)$ $\mathbf{M}_{\text{spin}}$ and the charge-$SU(2)$ $\mathbf{M}_{\text{charge}}$ are exchanged after the partial particle–hole transformation.

We are ready to verify the invariance of the transfer matrix of the 1D Hubbard model under the partial particle–hole transformation. First we assume that N is even and impose the periodic boundary condition. Then the partial particle–hole transformation of the monodromy matrix induced by (4.1) is

$$\hat{T}(\theta; c_{ms}, U) = \mathbf{V}^{-1}T(\theta; c_{ms}, U)\mathbf{V}. \tag{4.8}$$

From the relations

$$\mathrm{str}\hat{T}(\theta; c_{ms}, U) = \mathrm{str}\{\mathbf{V}^{-1}T(\theta; c_{ms}, U)\mathbf{V}\} = -\mathrm{str}T(\theta; c_{ms}, U)$$

and

$$\mathrm{str}\hat{T}(\theta; c_{ms}, U) = \mathrm{i}^N \mathrm{str}T(\theta; \hat{c}_{ms}, -U)$$

we obtain [27]

$$\mathrm{str}T(\theta; c_{ms}, U) = -\mathrm{i}^N \mathrm{str}T(\theta; \hat{c}_{ms}, -U). \tag{4.9}$$

This proves the invariance of the fermionic transfer matrix (up to sign) under the the partial particle–hole transformation (1.7). Note a relation

$$\mathrm{str}\{X(\theta)\mathbf{V}\} = -\mathrm{str}\{\mathbf{V}X(\theta)\} \tag{4.10}$$

which should be compared with (3.14).

We have a similar relation for $\mathbf{N}$ odd,

$$\mathrm{str}\{\mathbf{K}T(\theta; c_{ms}, U)\} = -\mathrm{i}^{N-1}\mathrm{str}\{\mathbf{K}T(\theta; \hat{c}_{ms}, -U)\} \tag{4.11}$$

where $\mathbf{K}$ is given by (3.21). In the derivation of (4.11), we have used the relation

$$\mathbf{K}\bar{\mathbf{V}}^{-1} = \mathrm{i}\mathbf{V}^{-1}\mathbf{K}.$$

Formula (4.11) indicates that the fermionic transfer matrix for N odd is also invariant under the partial particle–hole transformation (1.7) when we assume the twisted periodic boundary condition (3.22).

5. $SO(4)$ symmetry of the higher conserved currents

In section 3, we have shown the $SO(4)$ symmetry of the transfer matrix, which means that all the conserved currents of the 1D Hubbard model also have the $SO(4)$ symmetry. As will be seen, the $SO(4)$ symmetry of the conserved currents can be manifestly read out in terms of the Clifford algebra. Hereafter, for simplicity of explanation, we assume the number of sites is always even and impose the periodic boundary condition.

Although the graded Yang–Baxter relation ensures the existence of infinitely many higher conserved currents in involution, it is not an easy task to obtain their explicit forms from the transfer matrix [23]. To construct the higher conserved currents, we often use the boost operator [35, 36], which recursively produces the higher conserved currents. However, in the case of the 1D Hubbard model, the boost operator does not exist [37] and we have to resort to a more direct computation.

The first higher conserved current of the 1D Hubbard model was found by Shastry [19, 21] as

$$\begin{aligned} I^{(2)} = \mathrm{i}\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c_{m+2s}^{\dagger}c_{ms} - c_{ms}^{\dagger}c_{m+2s}) \\ -\mathrm{i}U\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c_{m+1s}^{\dagger}c_{ms} - c_{ms}^{\dagger}c_{m+1s})(n_{m+1,-s} + n_{m,-s} - 1). \end{aligned} \tag{5.1}$$

Subsequently, some higher conserved currents were obtained in a similar fashion [37–39]

$$\begin{aligned} I^{(3)} = -\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c_{m+3s}^{\dagger}c_{ms} + c_{ms}^{\dagger}c_{m+3s}) + U\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}[(c_{m+1s}^{\dagger}c_{m-1s} + c_{m-1s}^{\dagger}c_{m+1s}) \\ \times(n_{m+1,-s} + n_{m,-s} + n_{m-1,-s} - \tfrac{3}{2}) + (c_{m+1s}^{\dagger}c_{ms} - c_{ms}^{\dagger}c_{m+1s}) \\ \times(c_{m,-s}^{\dagger}c_{m-1,-s} - c_{m-1,-s}^{\dagger}c_{m,-s}) - (n_{m+1s} - \tfrac{1}{2})(n_{m,-s} - \tfrac{1}{2})] \end{aligned}$$

$$+U\sum_{m=1}^{N}[(c^{\dagger}_{m+1\uparrow}c_{m\uparrow}-c^{\dagger}_{m\uparrow}c_{m+1\uparrow})(c^{\dagger}_{m+1\downarrow}c_{m\downarrow}-c^{\dagger}_{m\downarrow}c_{m+1\downarrow})$$

$$-(n_{m\uparrow}-\tfrac{1}{2})(n_{m\downarrow}-\tfrac{1}{2})]-U^{2}\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c^{\dagger}_{m+1s}c_{ms}+c^{\dagger}_{ms}c_{m+1s})(n_{m,-s}-\tfrac{1}{2})$$

$$\times(n_{m+1,-s}-\tfrac{1}{2})-\tfrac{U^{3}}{4}\sum_{m=1}^{N}(n_{m\uparrow}-\tfrac{1}{2})(n_{m\downarrow}-\tfrac{1}{2}) \tag{5.2}$$

$$I^{(4)}=\mathrm{i}\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c^{\dagger}_{m+4s}c_{ms}-c^{\dagger}_{ms}c_{m+4s})-2\mathrm{i}U\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}\Bigg[(c^{\dagger}_{m+3s}c_{ms}-c^{\dagger}_{ms}c_{m+3s})$$

$$\times\sum_{k=m}^{m+3}(n_{k,-s}-\tfrac{1}{2})-(c^{\dagger}_{m+1s}c_{ms}-c^{\dagger}_{ms}c_{m+1s})\sum_{k=m-1}^{m+2}(n_{k,-s}-\tfrac{1}{2})$$

$$+(c^{\dagger}_{m+2s}c_{ms}+c^{\dagger}_{ms}c_{m+2s})\sum_{k=m-1}^{m+2}(c^{\dagger}_{k+1,-s}c_{k,-s}-c^{\dagger}_{k,-s}c_{k+1,-s})\Bigg]$$

$$+4\mathrm{i}U^{2}\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}\{(c^{\dagger}_{m+2s}c_{ms}-c^{\dagger}_{ms}c_{m+2s})[(n_{m,-s}-\tfrac{1}{2})(n_{m+1,-s}-\tfrac{1}{2})$$

$$+(n_{m,-s}-\tfrac{1}{2})(n_{m+2,-s}-\tfrac{1}{2})+(n_{m+1,-s}-\tfrac{1}{2})(n_{m+2,-s}-\tfrac{1}{2})]$$

$$+(c^{\dagger}_{m+1s}c_{ms}+c^{\dagger}_{ms}c_{m+1s})[(c^{\dagger}_{m-1,-s}c_{m,-s}-c^{\dagger}_{m,-s}c_{m+1,-s})(n_{m+1,-s}-\tfrac{1}{2})$$

$$+(c^{\dagger}_{m+1,-s}c_{m+2,-s}-c^{\dagger}_{m+2,-s}c_{m+1,-s})(n_{m,-s}-\tfrac{1}{2})]\}$$

$$+2\mathrm{i}U^{3}\sum_{m=1}^{N}\sum_{s=\uparrow\downarrow}(c^{\dagger}_{m+1s}c_{ms}-c^{\dagger}_{ms}c_{m+1s})(n_{m+1,-s}+n_{m,-s}-1). \tag{5.3}$$

These currents are embedded in the fermionic transfer matrix (2.20) and should be $SO(4)$ invariant from the result in the previous section. One can confirm the $SO(4)$ invariance of these currents using the transformation law of the fermion operators (3.12). In the following, we present a different approach: we shall rewrite these currents in manifestly $SO(4)$ invariant forms by using the Clifford algebra.

Define $\Gamma^{a}_{j}(j=1,\ldots,N,a=1,\ldots,4)$ by

$$\begin{aligned}\Gamma^{1}_{2n-1}&=c^{\dagger}_{2n-1\uparrow}+c_{2n-1\uparrow} &\qquad \Gamma^{2}_{2n-1}&=\mathrm{i}(c^{\dagger}_{2n-1\uparrow}-c_{2n-1\uparrow})\\ \Gamma^{3}_{2n-1}&=c^{\dagger}_{2n-1\downarrow}+c_{2n-1\downarrow} &\qquad \Gamma^{4}_{2n-1}&=\mathrm{i}(c^{\dagger}_{2n-1\downarrow}-c_{2n-1\downarrow})\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\Gamma^{1}_{2n}&=\mathrm{i}(c_{2n\uparrow}-c^{\dagger}_{2n\uparrow}) &\qquad \Gamma^{2}_{2n}&=c^{\dagger}_{2n\uparrow}+c_{2n\uparrow}\\ \Gamma^{3}_{2n}&=\mathrm{i}(c_{2n\downarrow}-c^{\dagger}_{2n\downarrow}) &\qquad \Gamma^{4}_{2n}&=c^{\dagger}_{2n\downarrow}+c_{2n\downarrow}\end{aligned} \tag{5.5}$$

where $n=1,\ldots,\frac{N}{2}$. Then the operators Γ^{a}_{j} satisfy the defining relations of the Clifford algebra [12, 40]

$$\{\Gamma^{a}_{j},\Gamma^{b}_{k}\}=2\delta_{jk}\delta^{ab} \qquad j,k=1,\ldots,N \qquad a,b=1,\ldots,4. \tag{5.6}$$

In terms of Γ^{a}_{j}, the $SO(4)$ rotation for the fermion operators (3.12) can be expressed simply as

$$\tilde{\Gamma}^{a}_{j}=\sum_{b=1}^{4}\mathbf{G}^{ab}\Gamma^{b}_{j} \qquad \mathbf{G}=(\mathbf{G}^{ab})\in SO(4). \tag{5.7}$$

The relation between the matrices $\mathbf{G}$ and $\mathbf{M}$ in section 3 is explicitly given by

$$\mathbf{G} = \mathbf{G}^{(1)}\mathbf{G}^{(2)}$$
$$\mathbf{G}^{(1)} = \begin{pmatrix} \xi_0 & \xi_1 & \xi_3 & -\xi_2 \\ -\xi_1 & \xi_0 & -\xi_2 & -\xi_3 \\ -\xi_3 & \xi_2 & \xi_0 & \xi_1 \\ \xi_2 & \xi_3 & -\xi_1 & \xi_0 \end{pmatrix} \qquad \mathbf{G}^{(2)} = \begin{pmatrix} \zeta_0 & \zeta_1 & -\zeta_3 & -\zeta_2 \\ -\zeta_1 & \zeta_0 & \zeta_2 & -\zeta_3 \\ \zeta_3 & -\zeta_2 & \zeta_0 & -\zeta_1 \\ \zeta_2 & \zeta_3 & \zeta_1 & \zeta_0 \end{pmatrix}$$

where ξ_i and η_i are real numbers given by

$$\begin{aligned} &\xi_0 = \mathrm{Re}\,(M_{11}) \qquad \xi_1 = \mathrm{Im}\,(M_{11}) \qquad \xi_2 = \mathrm{Re}\,(M_{14}) \qquad \xi_3 = \mathrm{Im}\,(M_{14}) \\ &\zeta_0 = \mathrm{Re}\,(M_{22}) \qquad \zeta_1 = \mathrm{Im}\,(M_{22}) \qquad \zeta_2 = \mathrm{Re}\,(M_{23}) \qquad \zeta_3 = \mathrm{Im}\,(M_{23}) \\ &\sum_{j=0}^{3} \xi_j^2 = \sum_{j=0}^{3} \zeta_j^2 = 1. \end{aligned} \tag{5.8}$$

Clearly the matrix $\mathbf{G}^{(1)}$ corresponds to the charge-$SU(2)$ and the matrix $\mathbf{G}^{(2)}$ corresponds to the spin-$SU(2)$. It is an interesting exercise to confirm that the matrices $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ commute each other

$$\mathbf{G}^{(1)}\mathbf{G}^{(2)} = \mathbf{G}^{(2)}\mathbf{G}^{(1)}.$$

Define the operator Γ_j^5 by

$$\Gamma_j^5 = \Gamma_j^1\Gamma_j^2\Gamma_j^3\Gamma_j^4 = \frac{1}{4!}\sum_{a,\ldots,d=1}^{4} \epsilon_{abcd}\Gamma_j^a\Gamma_j^b\Gamma_j^c\Gamma_j^d. \tag{5.9}$$

The operator Γ_j^5 has the following properties

$$\{\Gamma_j^5, \Gamma_j^a\} = 0 \qquad [\Gamma_j^5, \Gamma_k^a] = 0 \qquad j \neq k \qquad a = 1, \ldots, 4. \tag{5.10}$$

It is clear that the operators such as

$$\sum_{a=1}^{4} \Gamma_j^a\Gamma_k^a \qquad \Gamma_j^5 \tag{5.11}$$

are invariant under the $SO(4)$ rotation (5.7). We use this fact to rewrite the conserved currents. The Hamiltonian $\mathcal{H} = I^{(1)}$ in terms of the operators Γ_j^a and Γ_j^5 is [12, 40]

$$I^{(1)} = \sum_j \sum_a (-1)^j \Gamma_{j+1}^a \Gamma_j^a + u \sum_j \Gamma_j^5 \tag{5.12}$$

where

$$u = \frac{\mathrm{i}U}{2}.$$

Hereafter, we neglect the difference of an overall factor. Formula (5.12) gives a manifestly $SO(4)$ invariant representation of the Hamiltonian.

In the same way, we express the higher conserved currents in terms of the operators Γ_j^a and Γ_j^5 as

$$I^{(2)} = \sum_j \sum_a \Gamma_{j+2}^a \Gamma_j^a + u \sum_j \sum_a (-1)^j \Gamma_{j+1}^a \Gamma_j^a (\Gamma_{j+1}^5 - \Gamma_j^5) \tag{5.13}$$

$$I^{(3)} = \sum_j \sum_a (-1)^j \Gamma_{j+3}^a \Gamma_j^a - u \sum_j \Bigg[\Gamma_j^5 + \sum_a \Gamma_{j+2}^a \Gamma_j^a (\Gamma_{j+2}^5 - \Gamma_j^5)$$

$$+\sum_{a\neq b}(\Gamma^a_{j+2}\Gamma^a_{j+1}\Gamma^b_{j+1}\Gamma^b_j\Gamma^5_{j+1}-\tfrac{1}{2}\Gamma^a_{j+1}\Gamma^b_{j+1}\Gamma^a_j\Gamma^b_j\Gamma^5_j)\bigg]$$

$$+u^2\sum_j\sum_a(-1)^j\Gamma^a_{j+1}\Gamma^a_j\Gamma^5_{j+1}\Gamma^5_j+u^3\sum_j\Gamma^5_j \tag{5.14}$$

$$I^{(4)}=\sum_j\sum_a\Gamma^a_{j+4}\Gamma^a_j+u\sum_j(-1)^j\bigg\{\sum_a[\Gamma^a_{j+3}\Gamma^a_j(\Gamma^5_{j+3}-\Gamma^5_j)-\Gamma^a_{j+1}\Gamma^a_j(\Gamma^5_{j+1}-\Gamma^5_j)]$$

$$+\sum_{a\neq b}[\Gamma^a_{j+3}\Gamma^a_{j+2}\Gamma^b_{j+2}\Gamma^b_j\Gamma^5_{j+2}+\Gamma^a_{j+3}\Gamma^a_{j+1}\Gamma^b_{j+1}\Gamma^b_j\Gamma^5_{j+1}$$

$$-\Gamma^a_{j+2}\Gamma^b_{j+2}\Gamma^a_{j+1}\Gamma^b_j\Gamma^5_{j+2}+\Gamma^a_{j+2}\Gamma^b_{j+1}\Gamma^a_j\Gamma^b_j\Gamma^5_j]\bigg\}$$

$$+u^2\sum_j[\sum_a\Gamma^a_{j+2}\Gamma^a_j\Gamma^5_{j+2}\Gamma^5_j-\sum_{a\neq b}\Gamma^a_{j+2}\Gamma^a_{j+1}\Gamma^b_{j+1}\Gamma^b_j\Gamma^5_{j+1}(\Gamma^5_{j+2}-\Gamma^5_j)]$$

$$+u^3\sum_j\sum_a(-1)^j\Gamma^a_{j+1}\Gamma^a_j(\Gamma^5_{j+1}-\Gamma^5_j). \tag{5.15}$$

Since the terms that constitute (5.13)–(5.15) are of the form (5.11), we can see that the higher conserved currents $I^{(2)}$, $I^{(3)}$ and $I^{(4)}$ are also manifestly $SO(4)$ invariant. Note that the constraints $a\neq b$ on the summations do not break the $SO(4)$ symmetry. For example, we can write

$$\sum_{a\neq b}\Gamma^a_{j+2}\Gamma^a_{j+1}\Gamma^b_{j+1}\Gamma^b_j\Gamma^5_{j+1}=\sum_{a,b}\Gamma^a_{j+2}\Gamma^a_{j+1}\Gamma^b_{j+1}\Gamma^b_j\Gamma^5_{j+1}-\sum_a\Gamma^a_{j+2}\Gamma^a_j\Gamma^5_{j+1}. \tag{5.16}$$

Both terms on the RHS of (5.16) are clearly $SO(4)$ invariant.

The infinitesimal generators of the $SO(4)$ rotations (5.7) are given by [41]

$$Q^{ab}=-Q^{ba}=\frac{1}{4\mathrm{i}}\sum_j[\Gamma^a_j,\Gamma^b_j]. \tag{5.17}$$

In fact the generator (5.17) fulfils the defining relation of the Lie algebra $so(4)$

$$[Q^{ab},Q^{cd}]=-\mathrm{i}(\delta^{bc}Q^{ad}-\delta^{ac}Q^{bd}-\delta^{bd}Q^{ac}+\delta^{ad}Q^{bc}). \tag{5.18}$$

We also have a relation

$$[Q^{ab},\Gamma^c_j]=\mathrm{i}(\delta^{ac}\Gamma^b_j-\delta^{bc}\Gamma^a_j) \tag{5.19}$$

which is nothing but the infinitesimal transformation of (5.7). Using (5.19), one can confirm the commutativity

$$[Q^{ab},I^{(n)}]=0\qquad n=1,\ldots,4 \tag{5.20}$$

which shows the Lie algebra $so(4)$ symmetry of the conserved currents $I^{(n)}$. Actually the generators of the spin-$su(2)$ (1.5) and the charge-$su(2)$ (1.6) are related to Q^{ab} as

$$S^x=-\tfrac{1}{2}(Q^{14}-Q^{23})\qquad S^y=\tfrac{1}{2}(Q^{24}+Q^{13})\qquad S^z=-\tfrac{1}{2}(Q^{12}-Q^{34}) \tag{5.21}$$

$$\eta^x=-\tfrac{1}{2}(Q^{14}+Q^{23})\qquad \eta^y=\tfrac{1}{2}(Q^{24}-Q^{13})\qquad \eta^z=-\tfrac{1}{2}(Q^{12}+Q^{34}) \tag{5.22}$$

where we introduced S^x, S^y, η^x and η^y through the relations

$$S^{\pm}=S^x\pm\mathrm{i}S^y\qquad \eta^{\pm}=\eta^x\pm\mathrm{i}\eta^y.$$

For the Clifford algebra (5.6), the partial particle–hole transformation (1.7) corresponds to

$$\Gamma^3_j\longrightarrow-\Gamma^3_j. \tag{5.23}$$

Note that transformation (5.23) exchanges the spin-$su(2)$ (5.21) and the charge-$su(2)$ (5.22). As for the conserved currents, transformation (5.23) preserves the operators such as $\sum_{a=1}^{4} \Gamma_j^a \Gamma_k^a$, but changes the sign of Γ_j^5. From the explicit formulae (5.12)–(5.15), one can immediately find that the conserved currents $I^{(n)} (n = 1, \ldots, 4)$ are invariant under the partial particle–hole transformation

$$\Gamma_j^5 \longrightarrow -\Gamma_j^5 \qquad u \longrightarrow -u. \tag{5.24}$$

This is consistent with the result in section 4.

6. Discussions

We have investigated the $SO(4)$ symmetry of the 1D Hubbard model from the QISM point of view. Our approach is based on the fermionic formulation of the Yang–Baxter relation for the 1D Hubbard model found by Olmedilla *et al* [22]. It consists of the fermionic R-matrix and the fermionic L-operator. We have discovered the transformation law (3.9) of the fermionic L-operator under the $SO(4)$ rotation. It is a kind of gauge transformation and induces the transformation of the monodromy matrix. Using these properties, we have established the $SO(4)$ invariance of the transfer matrix. We have also discussed the case in which the number of lattice sites is odd. In this case, it is necessary to twist the periodic boundary condition.

Although the approach is different, our result can be considered as a Lie group generalization of the $su(2) \oplus su(2)$ symmetry in [27].

The $SO(4)$ symmetry will play an important role in the algebraic Bethe ansatz for the 1D Hubbard model, which was recently explored by Ramos and Martins [34]. In particular, we have clarified the transformation laws of the elements of the monodromy matrix under the $SO(4)$ rotation. They are used to construct the eigenstates of the transfer matrix.

We would like to emphasize the advantage of the fermionic formulation of the Yang–Baxter relation. It is difficult to discuss the $SO(4)$ symmetry of the Hubbard model through Shastry's R-matrix and the related transfer matrix.

The $SO(4)$ invariance of the transfer matrix ensures the $SO(4)$ invariance of the conserved currents. We have demonstrated the $SO(4)$ symmetry of the higher conserved currents employing the Clifford algebra, which corresponds to the spinor representation of the rotation group. It should be interesting to explore a representation of the fermionic L-operator itself in terms of the Clifford algebra.

On the infinite lattice, the Lie algebra $so(4) = su(2) \oplus su(2)$ symmetry of the 1D Hubbard model is extended to the Yangian $\mathrm{Y}(so(4)) = \mathrm{Y}(su(2)) \oplus \mathrm{Y}(su(2))$ symmetry [42, 43]

$$[\mathrm{Y}(so(4)), I^{(1)}] = 0$$

as was discovered by Uglov and Korepin [44]. The generators of $\mathrm{Y}(so(4))$ can be expressed in terms of the Clifford algebra Γ_j^a as follows

$$Q_{ab}^{(0)} = -\frac{\mathrm{i}}{4} \sum_j [\Gamma_j^a, \Gamma_j^b]$$

$$Q_{ab}^{(1)} = -\mathrm{i} \sum_j (-1)^j (\Gamma_{j+1}^a \Gamma_j^b + \Gamma_j^a \Gamma_{j+1}^b) + \frac{\mathrm{i}u}{4} \left(\sum_{j>k} - \sum_{k>j} \right) \sum_{c \neq a,b} \Gamma_j^a \Gamma_j^c \Gamma_k^c \Gamma_k^b (\Gamma_j^5 + \Gamma_k^5).$$

By using the fundamental properties of the Clifford algebra (5.6), (5.9) and (5.10), we have confirmed that the higher conserved currents $I^{(n)} (n = 2, 3, 4)$ also have the Yangian

$\mathrm{Y}(so(4))$ symmetry, i.e.

$$[Q_{ab}^{(0)}, I^{(n)}] = [Q_{ab}^{(1)}, I^{(n)}] = 0 \qquad n = 1, \ldots, 4.$$

All the conserved currents of the 1D Hubbard model on the infinite lattice are conjectured to have the $\mathrm{Y}(so(4))$ symmetry. In fact Murakami and Göhmann [45] recently showed the existence of an infinite number of the conserved currents which have the Yangian symmetry on the infinite lattice. However, one of the two $\mathrm{Y}(su(2))$ that constitute $\mathrm{Y}(so(4))$ drops out [45]. It seems to be difficult to prove the full $\mathrm{Y}(so(4))$ symmetry of the conserved currents simultaneously in their method.

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Appendix. Symmetry matrix of Shastry's R-matrix

The L-operator for the coupled spin model (1.11) [19–22] is expressed as

$$L_m(\theta) = \begin{pmatrix} \mathrm{e}^h p_m^+(\theta) q_m^+(\theta) & p_m^+(\theta)\tau_m^- & \sigma_m^- q_m^+(\theta) & \mathrm{e}^h \sigma_m^- \tau_m^- \\ p_m^+(\theta)\tau^+ & \mathrm{e}^{-h} p_m^+(\theta) q_m^-(\theta) & \mathrm{e}^{-h}\sigma_m^-\tau_m^+ & \sigma_m^- q_m^-(\theta) \\ \sigma_m^+ q_m^+(\theta) & \mathrm{e}^{-h}\sigma_m^+\tau_m^- & \mathrm{e}^{-h} p_m^-(\theta) q_m^+(\theta) & p_m^-(\theta)\tau_m^- \\ \mathrm{e}^h\sigma_m^+\tau_m^+ & \sigma_m^+ q_m^-(\theta) & p_m^-(\theta)\tau_m^+ & \mathrm{e}^h p_m^-(\theta) q_m^-(\theta) \end{pmatrix} \tag{A.1}$$

where

$$\begin{aligned} p_m^{\pm}(\theta) &= \tfrac{1}{2}(\cos\theta + \sin\theta) \pm \tfrac{1}{2}(\cos\theta - \sin\theta)\sigma_m^z \\ q_m^{\pm}(\theta) &= \tfrac{1}{2}(\cos\theta + \sin\theta) \pm \tfrac{1}{2}(\cos\theta - \sin\theta)\tau_m^z. \end{aligned} \tag{A.2}$$

Here the coupling constant h is related to the spectral parameter θ through formula (2.2). The R-matrix $\check{R}_{12}(\theta_1, \theta_2)$, which satisfies the Yang–Baxter relation (1.12) with the L-operator (A.1), is connected to the fermionic R-matrix (2.5) through the formula [22]

$$\check{R}_{12}(\theta_1, \theta_2) = W_{12}^{-1} \check{\mathcal{R}}_{12}(\theta_1, \theta_2) W_{12} \tag{A.3}$$

where W_{12} is a diagonal 16×16 matrix

$$W_{12} = \mathrm{diag}(1, 1, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, 1, 1, -1, -1, \mathrm{i}, \mathrm{i}, \mathrm{i}, \mathrm{i}, -1, -1). \tag{A.4}$$

We consider the symmetry matrix M of Shastry's R-matrix $\check{R}_{12}(\theta_1, \theta_2)$ which is defined to be a constant matrix satisfying

$$[\check{R}_{12}(\theta_1, \theta_2), \mathsf{M} \otimes \mathsf{M}] = 0. \tag{A.5}$$

Here the matrix elements of M are assumed to be commuting numbers. One might suppose the symmetry matrix of Shastry's R-matrix is identical to that of the fermionic R-matrix (2.14). However, surprisingly enough, we notice that they take different forms. In fact, solving the defining equation for the symmetry matrix (A.5) directly, we find that the followings are the symmetry matrix of Shastry's R-matrix

$$\mathsf{M} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \delta & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & \gamma & 0 & 0 \\ \delta & 0 & 0 & 0 \end{pmatrix} \tag{A.6}$$

where α, β, γ and δ are c-numbers obeying

$$\alpha\delta = \beta\gamma. \tag{A.7}$$

We ignore the difference of overall factors of the matrices. Then each matrix (A.6) depends only on two parameters. The result means that Shastry's R-matrix does not reflect the $SO(4)$ symmetry of the fermionic Hamiltonian (1.1). Therefore the $SO(4)$ symmetry of the transfer matrix that we explored in this paper may not be discussed if we use Shastry's R-matrix and L-operator (A.1).

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